JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 18, No. 5, September–October 1995

# Optimum Guidance with a Single Uncertain Time Lag

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**A disturbance attenuation problem that corresponds to optimal target interception is considered. In this problem the target's jerk is modeled as a finite energy disturbance, and the commanded acceleration of the interceptor is the control variable, to which the actual acceleration is assumed to be related via a single uncertain time lag. The interceptor's optimal strategy is obtained by considering an auxiliary disturbance attenuation problem in which an additional fictitious disturbance of finite energy, which corresponds to the uncertainty, is introduced. The resulting robust guidance law is compared to proportional navigation and minimum effort guidance (with zero miss constraint) for situations with and without uncertainty and in the presence of heading error and various target maneuvers. The robust guidance law turns out to be the best choice in the setup we have worked with, which is linear, continuous time, and assumes perfect measurements of relative position, velocity, and accelerations of the interceptor and the target.**

## Nomenclature

$N'$ = navigation constant
$t_{go}$ = time to go until intercept
$V_c$ = closing velocity
$y$ = relative interceptor–target separation
$()^T$ = transpose

## I. Introduction

THE method of proportional navigation (e.g., Ref. 1) is widely applied for interception of nonmaneuvering targets. In cases where the target maneuvers are significant, extensions of this method are applied such as augmented proportional navigation (see Ref. 2), where the commanded interceptor's acceleration $n_c$ depends on the target acceleration $n_T$, so that

$$n_c = \frac{N'(y + \dot{y} t_{go})}{t_{go}^2} + \frac{N'}{2} n_T \tag{1}$$

Note that the line-of-sight rate is given by

$$\dot{\lambda} = (y + \dot{y} t_{go}) / V_c (t_{go})^2 \tag{2}$$

Therefore, the first term of Eq. (1) can be easily mechanized given gimbaled seeker, which measures the line-of-sight rate $\dot{\lambda}$. For the second term, however, an estimation of the target acceleration $n_T$ is required. This is not a trivial task. Note that the estimation process involves a nonzero time lag, but its value is, in general, known and can be compensated for. If the transfer function that relates the commanded and actual accelerations $n_c$ and $n_L$ has a significant time lag (with respect to the final time $t_f$), however, the augmented proportional navigation law can lead to a significant miss distance. To compensate for this, one can employ the minimum effort guidance law, which minimizes

$$J = \int_0^{t_f} n_c^2(t)\,dt$$

subject to $y(t_f) = 0$. The resulting guidance law nicely overcomes the large time-lag problem, but it is strongly dependent on the time constant and the time to go, namely,

$$n_c = \frac{N'(y + \dot{y} t_{go})}{t_{go}^2} + \frac{N'}{2} n_T - n_L K_L \tag{3}$$

where

$$N' = \frac{6h^2(e^{-h} - 1 + h)}{2h^3 + 3 + 6h - 6h^2 - 12he^{-h} - 3e^{-2h}} \tag{4}$$

and

$$K_L = \tau^2(e^{-h} + h - 1) \tag{5}$$

and where

$$h = t_{go}/\tau \tag{6}$$

In these equations $\tau$ is the time constant that relates the actual interceptor's acceleration and the corresponding command, namely,

$$n_L/n_c = 1/(1 + \tau s) \tag{7}$$

where $s$ is the differentiation operator, namely, $s = d/dt$.

A practical question to ask is, what happens if $\tau$ is not perfectly known, and how the uncertainty in $\tau$ affects the performance of the guidance law of Eqs. (3–6). The underlying assumption of this guidance law concerning the constant speeds of the adversaries is valid along the pursuit phase, but the pursuer can have very different speeds in different conflicts, depending on the time from launch to target acquisition and on the flight conditions. This can cause a significant uncertainty in the estimated value of $\tau$.

A similar question can be asked about the effect of the estimation error of the time to go. It has been shown in Ref. 5 that such an estimation error can significantly degrade the performance of the optimal guidance law. This problem, however, is not considered in the present work, and it is not clear yet whether the approach taken hereafter is adequate for dealing with this type of uncertainty. It should also be noted that this research is confined to the control part of the guidance problem whereas the very important issues of the estimation part are not discussed here.

The main contribution of this work is the novel control policy obtained from a formulation of an optimal interception problem that takes into account the uncertainty of the time constant $\tau$. Our problem is, in fact, a disturbance attenuation problem, where the target jerk is modeled as a finite energy disturbance and where a finite energy commanded acceleration has to be found so that the effect of the disturbance on a quadratic functional of the miss distance and the control effort are minimized. Our problem definition nicely fits into the framework of finite time horizon $H_\infty$-optimal control,[6–9] which deals with real bounded time-varying uncertainties.[10]

## II. Problem Formulation

We shall make the following assumptions.

1) The pursuit-evasion conflict is two dimensional, in the horizontal plane.

Received Oct. 21, 1994; revision received April 19, 1995; accepted for publication April 21, 1995. 

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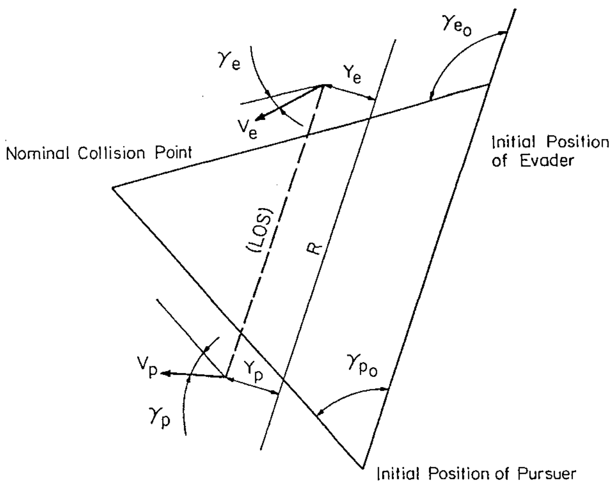


**Fig. 1 Interception conflict geometry.**

2) The speeds of the pursuer $P$ and the evader $E$ are constant.

*Remark 1*: This assumption is valid in the course of the conflict only. The missile itself, however, is assumed to have different speeds for different conflicts, depending on the elapsed time from launch to target acquisition.

3) The trajectories of $P$ and $E$ can be linearized around their collision course.

4) The evader can directly control its lateral jerk (acceleration's rate of change).

5) The pursuer controls its lateral acceleration via a first-order system.

*Remark 2*: This first-order system represents a closed-loop control of the missile's acceleration. It follows from the preceding remark that the time constant of this representation can vary with the speed of the missile at the acquisition time. This is typical for missiles with no measurements adequate for gain scheduling (e.g., pitot sensors). In these cases variations in the time constant can be quite significant.

Referring to Fig. 1 we obtain the following equation. For small $\gamma_p$ and $\gamma_e$,

$$\dot{Y} = \dot{Y}_e - \dot{Y}_p = V_e \cos(\gamma_{e0})\gamma_e - V_p \cos(\gamma_{p0})\gamma_p \tag{8}$$

This is obtained by noting that for $|\gamma_p| \ll 1$ we have

$$\sin(\gamma_{p_0} + \gamma_p) \approx \sin(\gamma_{p0}) + \cos(\gamma_{p0})\gamma_p$$

A similar condition holds for the evader $E$. We have also used the fact that the collision condition is expressed as

$$V_p \sin(\gamma_{p0}) - V_e \sin(\gamma_{e0}) = 0$$

Since the controls are the lateral accelerations, we define

$$x_1 \equiv -Y, \qquad x_2 \equiv -\dot{Y}, \qquad x_3 \equiv V_p \cos(\gamma_{p0})\dot{\gamma}_p$$

$$x_4 \equiv V_e \cos(\gamma_{e0})\dot{\gamma}_e$$

and readily obtain the following state space formulation:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) - x_4(t) \\ \dot{x}_3(t) &= -a(t)x_3(t) + a(t)u(t) \\ \dot{x}_4(t) &= w(t) \end{aligned} \tag{9}$$

where $w(t)$ is the evader's jerk and $u(t)$ is the commanded interceptor's acceleration.

The time constant

$$\tau(t) = 1/a(t) \tag{10}$$

is assumed to be in a prescribed interval, that is,

$$0 < a_0 \le a(t) \le a_1 \tag{11}$$

so that the interceptor dynamics are open-loop stable. We also assume that $w(t)$ is of finite energy, that is,

$$\int_0^{t_f} w^2(t)\,\mathrm{d}t < \infty \tag{12}$$

We note that the initial conditions for Eq. (9) are not necessarily zero, namely,

$$x(0) \equiv [x_1(0), x_2(0), x_3(0), x_4(0)] \equiv x_0 \neq 0 \tag{13}$$

Therefore, a variety of errors that may contribute to nonzero miss distance are allowed by Eqs. (12) and (13), for example, those listed as follows.

1) Heading error is obtained by taking $x_2(0) = V_c\chi_e$ where $V_c = V_p\cos(\gamma_{p_0}) - V_e\cos(\gamma_{e_0})$ is the closing velocity and $\chi_e$ is the heading error.

2) Constant target acceleration is obtained by taking $x_4(0) = n_T = \text{const}$ and $w(t) = 0$.

3) Ramp in target acceleration is obtained by taking $w(t) = \text{const}$.

4) Step in target's acceleration is obtained by taking $w(t)$ to be a finite energy approximation to $\delta(t - t_0)$, where $t_0$ is the instant of the step occurrence.

5) First-order Markov process for target's acceleration is obtained by taking $w(t) = -bx_4(t) + bv(t)$, where $v(t)$ is a white noise process of zero mean and $b$ is the process bandwidth. Note that for $b > 0$, $w(t)$ satisfies Eq. (12).

We define the following cost function:

$$J = x^T(t_f)Q_f x(t_f) + \int_0^{t_f} [u^2(t) - \gamma^2 w^2(t)]\,\mathrm{d}t \tag{14}$$

where

$$Q_f = \mathrm{diag}\{q, 0, 0, 0\}, \qquad q > 0 \tag{15}$$

Our problem is to find a finite energy $u(t)$ that utilizes a full-state feedback strategy, so that

$$J \le 0 \tag{16}$$

for the minimum possible $\gamma = \gamma_0$ and for all $a(t)$ that satisfies Eq. (11).

*Remark 3*: Minimization of $\gamma$ so that Eq. (16) is satisfied corresponds to minimization of the effect of the target's jerk on a weighted combination of the miss distance and the control effort. This fact can be better understood if Eqs. (14–16) are written for nontrivial $w(t)$ as

$$\left[qy^2(t_f) + \int_0^{t_f} u^2(t)\,\mathrm{d}t\right] \Big/ \int_0^{t_f} w^2(t)\,\mathrm{d}t \le \gamma^2 \to \min \tag{17}$$

*Remark 4*: If $q$ is taken to be very large, then the practical effect for this choice is a small miss distance, which will cost large $u(t)$.

*Remark 5*: Note that the disturbance attenuation problem of Eqs. (14–16) for $a_0 = a_1 = a(t)$ is one part of a linear-quadratic differential game[9] where a saddle-point relation can be satisfied under certain conditions. In this game, $w(t)$ maximizes $J$ and $u(t)$ minimizes $J$ so that

$$J(u^*, w) \le J(u^*, w^*) \le J(u, w^*) \tag{18}$$

This game is a zero sum game where $J(u^*, w^*) = 0$ (see Ref. 9) and, thus, the left inequality is equivalent to Eq. (16).

*Remark 6*: Note that our formulation allows closed-loop strategy for $w(t)$. This is why example 5 is possible in our setup.

The formulated problem has no closed-form solution for uncertain $a(t)$ in terms of a necessary and sufficient condition that ensures Eq. (16). We, therefore, use the method of Ref. 10 that gives only a sufficient condition and a rather simple solution for $u(t)$. This method applies an auxiliary disturbance attenuation problem that is defined (and related to our problem) in the next section.

## III. Auxiliary Problem

We denote the following:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \end{bmatrix} \tag{19}$$

The system of Eq. (9) is then given by

$$\dot{x} = Ax + B_1 w + B_2 u \tag{20}$$

where $A$ has an uncertain parameter. Together with the cost function of Eq. (12) our problem is a disturbance attenuation with an uncertain plant. Using the methods of Ref. 10 we can formulate an auxiliary problem for a completely known system where the uncertainty is represented by a corresponding finite energy fictitious disturbance. To this end, we define

$$\delta \equiv (a_1 - a_0)/2 \tag{21}$$

and let $\Delta$ to be a scalar with $|\Delta| < 1$ and also bring in a positive scalar scaling $\varepsilon$. This scalar has no effect on the problem formulation, but it is going to affect the matrices $L_a$ and $R_a$, which will be defined subsequently. The importance of this scaling idea will be discussed in the next section.

We represent $A$ as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon \\ 0 \end{bmatrix} \Delta [0 \quad 0 \quad \delta\varepsilon \quad 0] \tag{22}$$

where

$$\alpha = -[(a_1 + a_0)/2] \tag{23}$$

We also represent $B_2$ by

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon \\ 0 \end{bmatrix} \Delta\delta\varepsilon \tag{24}$$

We denote

$$L_a = \begin{bmatrix} 0 \\ 0 \\ 1/\varepsilon \\ 0 \end{bmatrix}, \qquad R_a = [0 \quad 0 \quad \varepsilon\delta \quad 0], \qquad R_b = \varepsilon\delta \tag{25}$$

and get the following representation of the system of Eq. (20):

$$\dot{x} = (A_\alpha + L_a \Delta R_a)x + B_1 w + (B_{2\alpha} + L_a \Delta R_b)u \tag{26}$$

where $A_\alpha$ and $B_\alpha$ are the system matrices with the nominal value $\alpha$.

We want to find a state-feedback controller $u(t)$ that makes $J$ of Eq. (14) negative for all $|\Delta| \le 1$. We apply the following result.[10]

*Theorem*: Given the following system:

$$\dot{x} = (A_\alpha + L_a \Delta R_a)x + B_1 w + (B_{2\alpha} + L_a \Delta R_b)u \tag{27}$$

$$z = C_1 x + D_{12} u \tag{28}$$

and the cost function

$$J = x^T(t_f) Q_f x(t_f) + \int_0^{t_f} (z^T z - \gamma^2 w^T w)\, \mathrm{d}t \tag{29}$$

then $J \le 0$ for all $\Delta^T \Delta \le I$, if the disturbance attenuation problem, which is associated with the following system:

$$\dot{x} = A_\alpha x + \begin{bmatrix} L_a & \gamma^{-1} B_1 \end{bmatrix} \tilde{w} + B_{2\alpha} u \tag{30}$$

$$\tilde{z} = \begin{bmatrix} R_a \\ C_1 \end{bmatrix} x + \begin{bmatrix} R_b \\ D_{12} \end{bmatrix} u \tag{31}$$

is solvable, in the sense that $J' \le 0$ where

$$J' = x^T(t_f) Q_f x(t_f) + \int_0^{t_f} (\tilde{z}^T \tilde{z} - \gamma^2 \tilde{w}^T \tilde{w})\, \mathrm{d}t \tag{32}$$

## IV. Solution of the Auxiliary Problem

We frame the following definitions:

$$\tilde{B}_1 = \begin{bmatrix} L_a & \gamma^{-1} B_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/\varepsilon & 0 \\ 0 & 1/\gamma \end{bmatrix} \tag{33}$$

$$\tilde{C}_1 = \begin{bmatrix} R_a \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \delta\varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$\tilde{D}_{12} = \begin{bmatrix} R_b \\ D_{12} \end{bmatrix} = \begin{bmatrix} \delta\varepsilon \\ 1 \end{bmatrix} \tag{35}$$

We note now that

$$R = \tilde{D}_{12}^T \tilde{D}_{12} = [\delta\varepsilon \quad 1][\delta\varepsilon \quad 1]^T = 1 + \delta^2\varepsilon^2 > 0 \tag{36}$$

and

$$\tilde{D}_{12}^T \tilde{C}_1 = [\delta\varepsilon \quad 1] \begin{bmatrix} 0 & 0 & \delta\varepsilon & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [0 \quad 0 \quad \delta^2\varepsilon^2 \quad 0] \tag{37}$$

The solution for this problem is given in Refs. 6 and 10. Let $P$ be the solution of the following matrix differential Riccati equation:

$$-\dot{P} = A_\alpha^T P + P A_\alpha - \left(P B_{2\alpha}^T + \tilde{C}_1^T \tilde{D}_{12}\right) R^{-1} \times \left(B_{2\alpha} P + \tilde{D}_{12}^T \tilde{C}_1\right) + P \tilde{B}_1 \tilde{B}_1^T P + \tilde{C}_1^T \tilde{C}_1 \tag{38}$$

$$P(t_f) = Q_f \tag{39}$$

It is obtained in Ref. 6 that $J' \le 0$ if and only if there exists $P$ that solves in the whole time interval $t \in [0, t_f]$. If such $P$ exists, the solution for $u(t)$ is given by

$$u(t) = -B_{2\alpha}^T P R^{-1} x \tag{40}$$

and the worst-case disturbance $w(t)$ is given by

$$w(t) = \tilde{B}_1^T P R^{-1} x \tag{41}$$

We note that $u(t)$ and $w(t)$ form a saddle-point pair for $J'$ of Eq. (29). Moreover, by the given theorem, the control strategy $u(t)$ ensures $J \le 0$. It does not, however, mean that the original cost function $J$ admits a saddle point!

We also note that by varying $\varepsilon$ we obtain a family of solutions, rather than a single one, all of which satisfy the formulated sufficiency condition for $J \le 0$. A one-dimensional search can be carried out in order to select a favorable one.

## V. Numerical Example

### A. Interception Conflict Description, Guidance Laws, Error Sources, and Uncertainties

To illustrate the merits of the robust guidance scheme we consider a numerical example, which is taken from Ref. 2. We analyze the effects on the miss distances of three types of target maneuvers, namely, a step in target acceleration of 3 $g$, a ramp in target acceleration of 1 $g$/s, and a sinusoidal target acceleration with frequency of 1 rad/s and magnitude of 3 $g$. We also analyze the influence of an initial heading error of 1 deg on the miss distance.

How these error sources affect the miss distance will be examined for the following three guidance laws: proportional navigation (with $N' = 4$), minimum effort guidance, and the robust guidance law of the present paper.

The design value for the speed of the interceptor is taken to be 1000 ft/s, corresponding to a design value of the airframe time constant of $\tau = 1$ s.

It is assumed that the airframe of the interceptor represents, in fact, the closed-loop transfer function of a normal acceleration control loop that does not utilize gain scheduling. Its time constant varies, therefore, along with the initial speed of the interceptor, which stays constant throughout the interception conflict. The dependence of the time constant $\tau$ on the speed $V_c$ is assumed to be of the type $\tau \sim 1/V_c^2$ (assuming $V_p \gg V_e$).

We consider two cases of interception. 1) The interceptor speed and the airframe time constant maintain their design values throughout, namely, 1000 ft/s and 1 s, respectively. 2) The interceptor speed and, consequently, the airframe time constant differ from their design values and are 500 ft/s and 4 s, respectively.

The large uncertainty we assume is caused by a combination of large speed variations at the beginning of the conflict and a lack of any kind of compensation within the acceleration loop for the flight condition changes. We have deliberately chosen an exaggerated example in order to emphasize the merits of the robust design. Clearly, when this uncertainty diminishes, our approach is advantageous only against maneuverable targets.

To enable a reasonable comparison between these two cases, we have assumed, in both cases, that the interception conflict duration is 10 s, i.e., $t_f = 10$ s, so that the initial range to go varies.

### B. Method of Simulation and State-Feedback Gains Computation

To solve this example, we have used a Fortran program that includes both adjoint and forward simulations of the dynamics of Eq. (9). The adjoint simulation model is identical to the model of Fig. 7.17 in Ref. 2. We note that the three guidance laws can, in fact, be represented by a state-feedback scheme like Eq. (40) where the guidance laws differ by the state feedback gain matrix [see Eqs. (1) and (3)].

The simulation first computes the miss distance due to step and ramp in target acceleration and heading error by solving the adjoint model numerically. The Riccati equation (38) is solved at this stage by changing the free variable from $t$ to $t_{go} = t_f - t$. Equation (38) is then solved from $t_{go} = 0$ to $t_{go} = t_f$ by taking the initial condition $P(t_{go} = 0) = Q_f$. The state feedback gains are computed according to

$$K(t_{go}) = -B_{2\alpha}^T P(t_{go}) R^{-1}$$

These gains are stored and used later in the forward simulation, by reversing the arrays in which the gains are stored. The forward simulation includes a constant target acceleration of 3 $g$, from which we want to learn about the values of the commanded acceleration on the various cases. The case of sinusoidal target maneuver is tested only by using the forward simulation. The simulation does not include any kind of nonlinearities.

In fact, four parameters affect the state-feedback gains that have to be chosen. These parameters are $\alpha$, $\gamma$, $\delta$, and $\varepsilon$. The value of $\alpha$ is taken as $-1$ rad/s, which corresponds to the design time constant of 1 s. The value of $\delta = 0.75$ corresponds, according to Eq. (22), to a range of airframe time constants between 4 s and about 0.6 s. The parameter $\varepsilon$, which serves as a scaling parameter, affects the state-feedback gains via Eqs. (25) and (38) and can be utilized to reach the minimum value of the minimum disturbance attenuation constant $\gamma$ for which Eq. (38) has still a solution. We have chosen $\varepsilon = 2$, which leads to $\gamma = \gamma_0 = 18$ (the minimum of $\gamma$ for which the escape time of the Riccati equation (38) is not in $t \in [0, t_f]$). Finally, the disturbance attenuation constant $\gamma$, which has to be greater than $\gamma = \gamma_0 = 18$, makes an important design tuning parameter. For the lowest possible $\gamma$ the guidance loop's bandwidth will be the largest, thus giving the best attenuation of the effect of the target jerk on the miss distance and control effort. As $\gamma$ is taken to infinity, less and less requirements are imposed on the target jerk. As a consequence, we obtain the lowest guidance-loop bandwidth. Taking $\gamma$ to be somewhere between these extremes is probably the right choice to make when the effect of target jerks as well as other error sources (e.g., heading error) are to be minimized. In our scan of admissible values of $\gamma$, we have found that $\gamma = 400$ is a good choice, which yields satisfactory results for all error sources (reasonably weighting the effect of target jerks with the other error sources), without applying excessive acceleration demands on the interceptor. Therefore, in the forthcoming discussions, the robust-guidance law was obtained by $\gamma = 400$.

### C. Simulation Results

To simplify the comparison, we classify the simulation results according to the error source (i.e. target acceleration step, heading error, etc.).

#### 1. *Step of 3 g in Target Acceleration*

The simulation results for a step of 3-$g$ acceleration for various $t_{go}$ are depicted in Figs. 2a, 2b, 3a, and 3b.

In the case where the actual time constant and the design time constant are identical (both 1 s), minimum-effort guidance is superior to robust guidance, in the miss distance (Fig. 2a) and, even more so, in the commanded interceptor acceleration (Fig. 2b). The proportional navigation results are comparable to the results of the minimum-effort and robust guidance laws when $t_{go}$ are larger than 7 s. The superiority of the minimum effort guidance law over the robust guidance law in the case with no uncertainty is the price we pay for having a robust guidance law.

For the case where the actual time constant is 4 s while the design time constant is still 1 s, the favorable guidance law is certainly the robust guidance law. The superiority of the robust guidance law over the minimum-effort law is both in the miss distance (Fig. 3a) and in the commanded accelerations (Fig. 3b). The proportional navigation law could not be expected to perform well for this large $\tau/t_{go}$ ratio and was included only for the sake of completeness.

*Remark 7*: The miss-distance results of the proportional navigation law are scaled in Figs. 2a and 3a by 10 and 50, respectively, in order to enable them to be shown on the same plots with the minimum-effort and robust guidance laws.

#### 2. *Ramp of 1 g/s in Target Acceleration*

For a ramp type target acceleration, the miss-distance results, as a function of the time to go when the maneuver starts, are depicted in Fig. 4a (the case where both the design and actual time constants are 1 s) and Fig. 4b (the case where the actual time constant is 4 s while the design time constant is still 1 s). The conclusions of Sec. C.1 also hold here. The minimum-effort guidance law, which is still superior when no uncertainty is present, fails to remain indifferent to the uncertainty as does the robust guidance law.

#### 3. *Heading Error of 1 deg*

For a heading error of 1 deg, the miss distance as function of the time to go is depicted in Fig. 5a (the case where both the design and actual time constants are 1 s) and Fig. 5b (the case where the actual time constant is 4 s while the design time constant is still 1 s). With this error source, the proportional navigation law does much better than with the previous error sources but is still hardly comparable to the minimum-effort and robust guidance laws. In this case as well, the minimum-effort guidance law does slightly better than the robust guidance law when no uncertainty exists (Fig. 5a) but performs worse than the latter with uncertainty present (Fig. 5b).

#### 4. *Sinusoidal Target Acceleration of 3 g with 1-rad/s Frequency*

The case of sinusoidal target acceleration is the only one where we see the obvious superiority of the robust guidance law over the minimum-effort guidance law, not only when uncertainty is present but also when there is no uncertainty. The results for this type of target maneuver were analyzed using only forward simulations, and we looked at two interesting features, namely, the normal separation ($Y$) between the interceptor and the target (Fig. 6a where both design and actual time constants are 1 s, and Fig. 7a where the actual time constant is 4 s while the design time constant remains at 1 s), and the commanded interceptor acceleration (Fig. 6b where both design and actual time constants are 1 s, and Fig. 7b where the actual time-constant is 4 s while the design time-constant remains at 1 s).

In Fig. 6a (no uncertainty) we see a very small interceptor–target normal separation, for the robust guidance, throughout the interception conflict (the miss distances with both guidance laws are below 0.5 ft), with no apparent cost in applying interceptor maneuvers

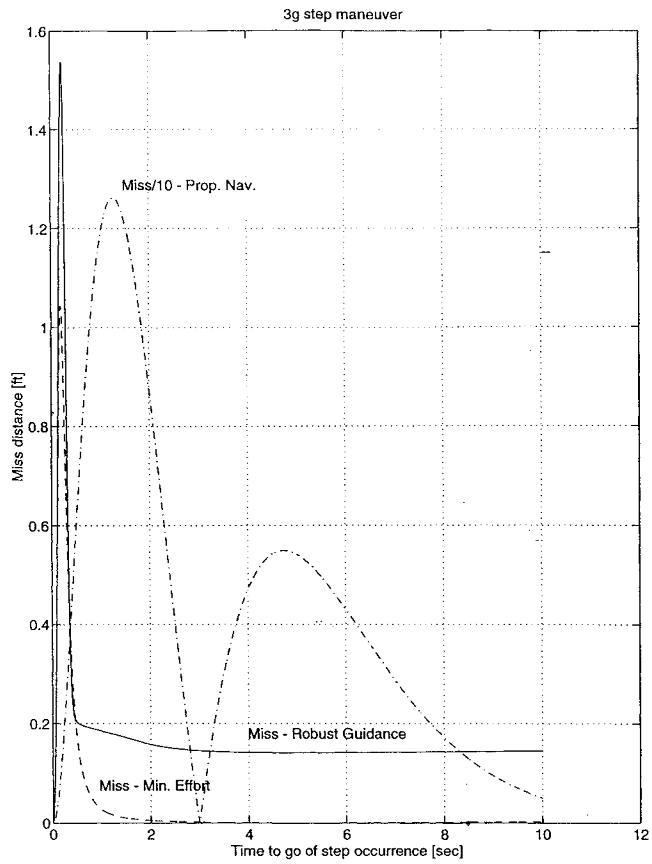


**Fig. 2a Miss distance due to a step of $3\,g$ in target acceleration, identical design and actual time constants.**

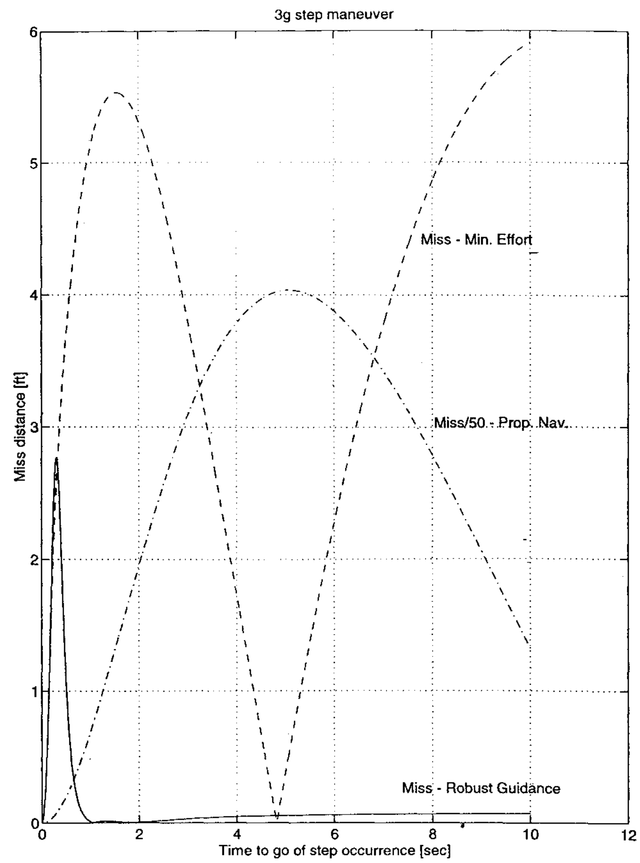


**Fig. 3a Miss distance due to a step of $3\,g$ in target acceleration, different design and actual time constants.**

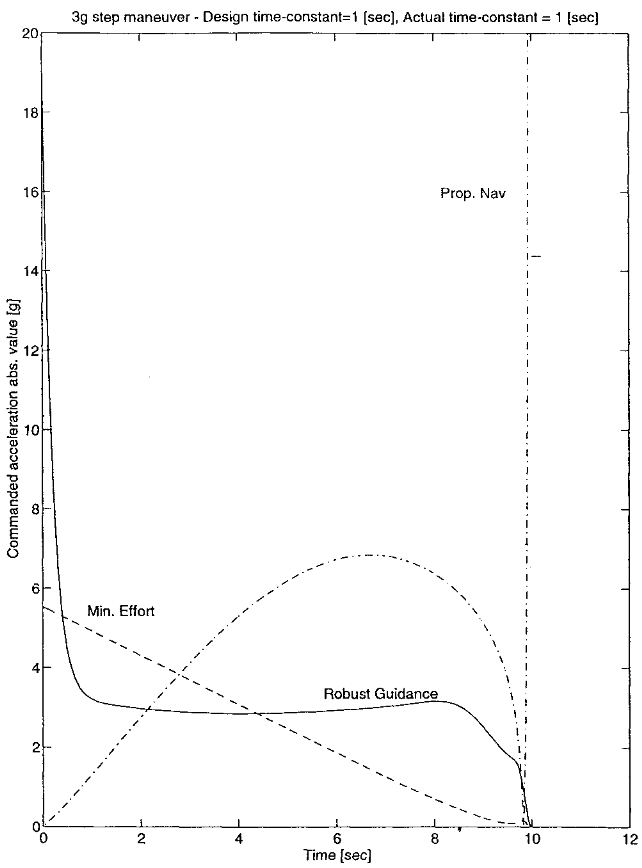


**Fig. 2b Commanded accelerations due to a step of $3\,g$ in target acceleration, identical design and actual time constants.**

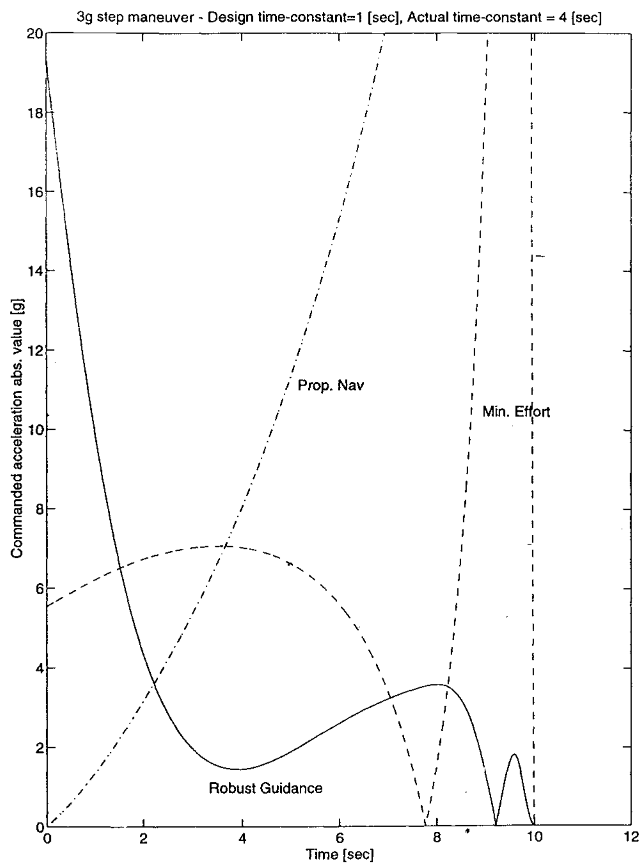


**Fig. 3b Commanded accelerations due to a step of $3\,g$ in target acceleration, different design and actual time constants.**

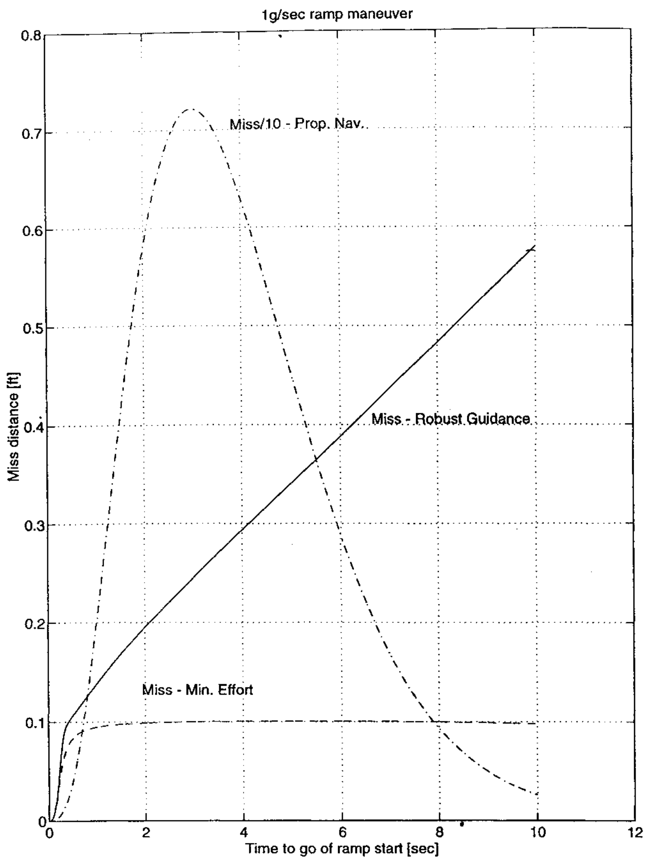


**Fig. 4a Miss distance due to a ramp of 1 *g*/s in target acceleration, identical design and actual time constants.**

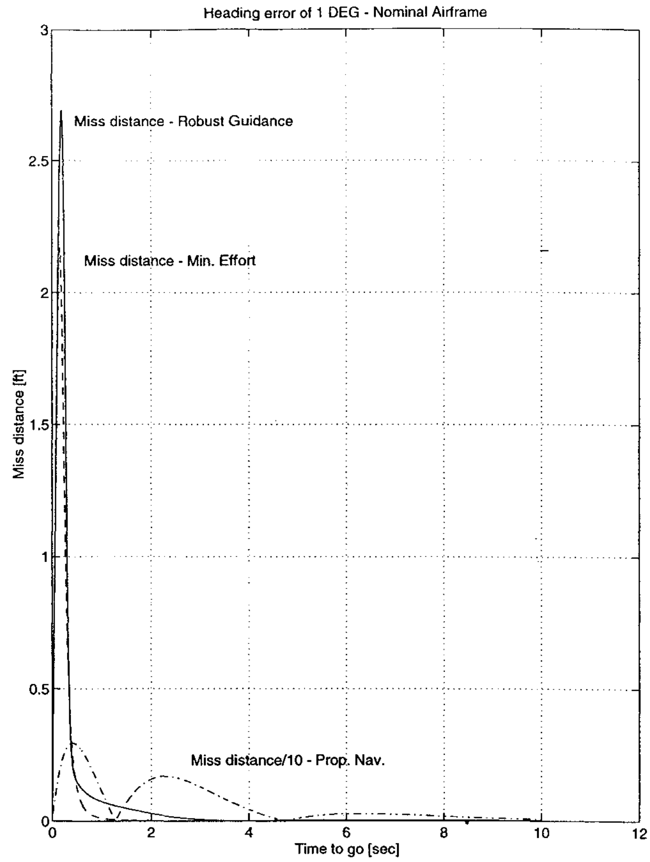


**Fig. 5a Miss distance due to a heading error of 1 deg, identical design and actual time constants.**

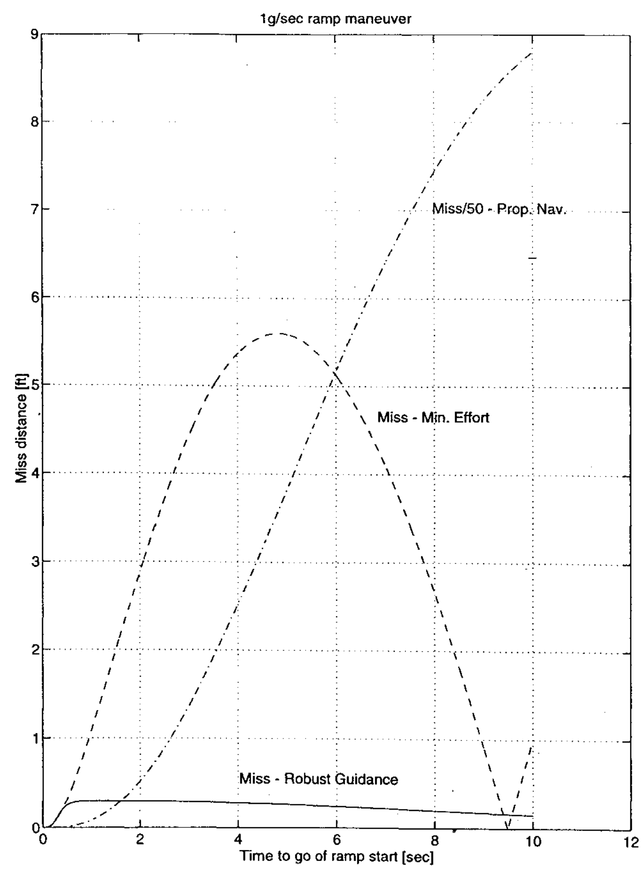


**Fig. 4b Miss distance due to a ramp of 1 *g*/s in target acceleration, different design and actual time constants.**

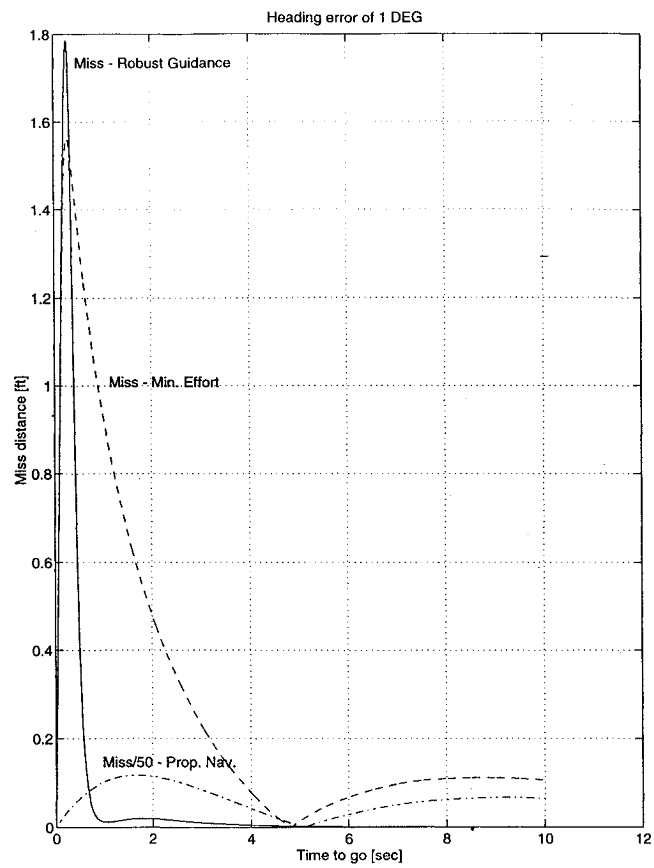


**Fig. 5b Miss distance due to a heading error of 1 deg, different design and actual time constants.**

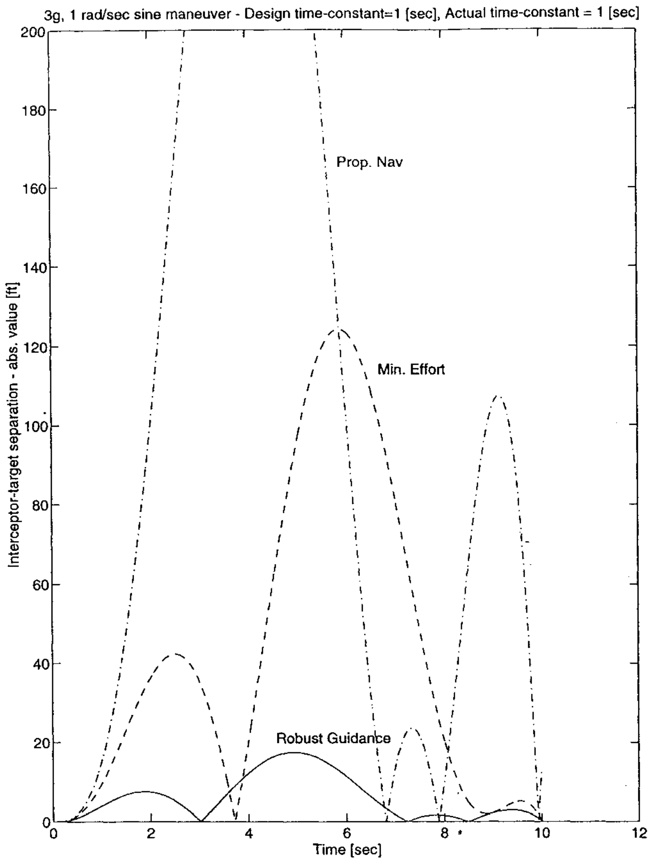


Fig. 6a **Interceptor–target normal separation due to a sine maneuver identical design and actual time constants.**

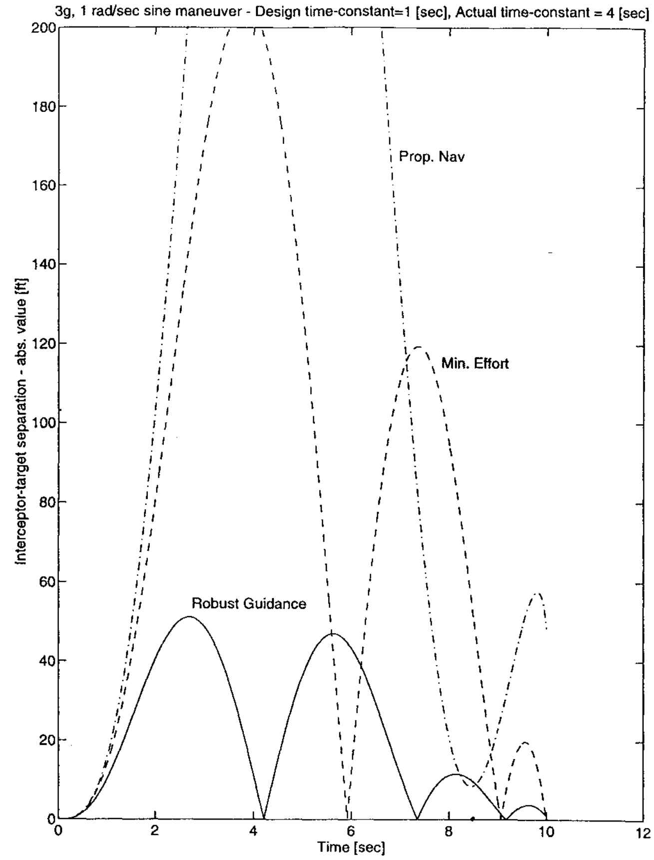


**Fig. 7a Interceptor–target normal separation due to a sine maneuver different design and actual time constants.**

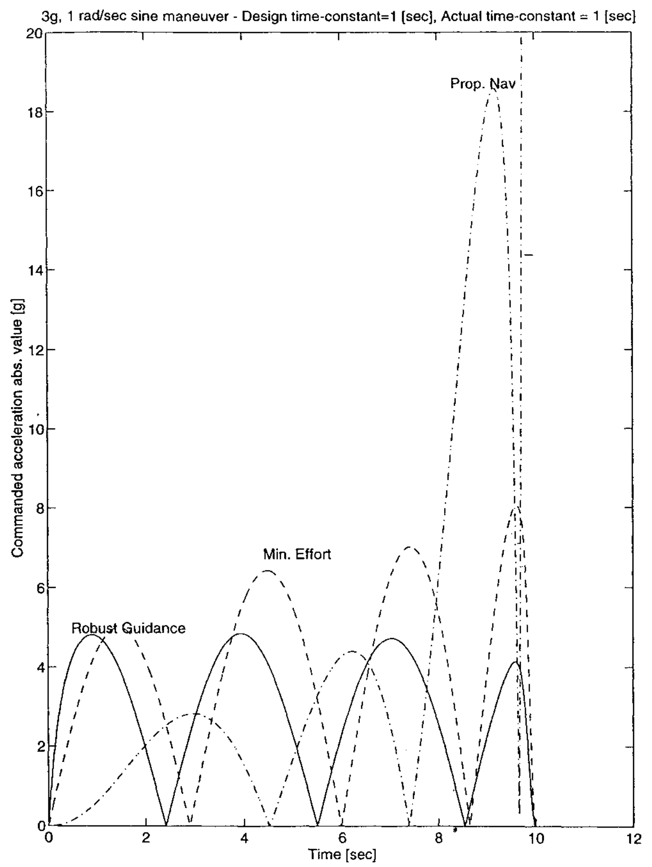


**Fig. 6b Commanded accelerations due to a sine maneuver, identical design and actual time constants.**

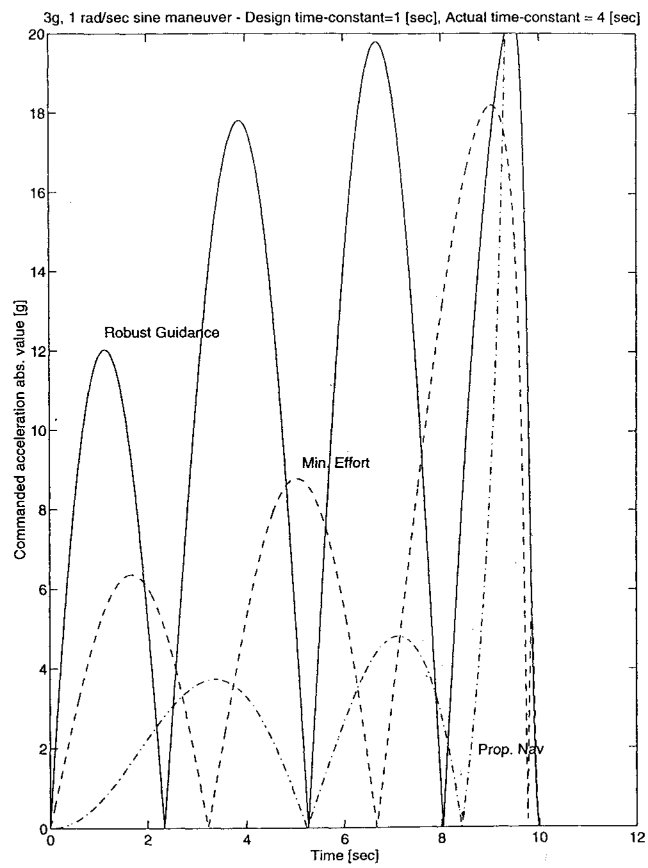


**Fig. 7b Commanded accelerations due to a sine maneuver, different design and actual time constants.**

(Fig. 6b). The difference is probably due to improved timing of the maneuvers. In this sense, therefore, the robust guidance law (being a game–theoretic solution that takes into account evasive maneuvers) is better than the others even for the designed system!

In Fig. 7a (uncertainty present) a larger interceptor–target normal separation is observed, but the robust guidance law is still clearly better than the others (the miss distance for the robust guidance is about 1 ft, whereas for the minimum effort, about 0.5 ft). The cost paid for the superiority of the robust guidance law (in terms of interceptor–target separation) is the larger interceptor commanded accelerations (Fig. 7b).

## VI. Conclusions

A robust guidance law that takes into account the uncertainty in the single time constant of the airframe and target evasive maneuvers was developed. The derivation applied the recently developed theory[10] of finite time horizon disturbance attenuation in the presence of real-bounded time-varying uncertainties.

The solution obtained is sufficient but not necessary; hence, a better guidance law may exist that our approach has not revealed. For the case without uncertainty, however, the solution is both necessary and sufficient and, therefore, optimal with respect to the given cost.

The resulting guidance policy utilizes a full state feedback, which can not be mechanized as easily as proportional navigation and its derivatives. The realization issues and the corresponding estimation problems are out of the scope of the present paper. It should be noted, however, that given a radar-based gimbaled seeker, the most difficult state to estimate is probably the target acceleration. In this context, the comparison in the present paper between the minimum-effort and the robust guidance laws can be considered to be a fair one, since the minimum-effort guidance law also requires the measurement of target acceleration. The issue of estimating target acceleration is thoroughly discussed in Ref. 12 and the references therein.

The results of this paper show, therefore, that some price has to be paid in order to achieve robustness. In our case the penalty we pay is twofold: the performance when no uncertainty exists is somewhat degraded and the realization of the guidance law becomes more difficult.

It has been demonstrated that one can employ $\gamma$ as a design parameter, indicating the relative importance of the targets' maneuvers. Thus, small $\gamma$ indicate a highly maneuverable evader whereas large $\gamma$ indicate the opposite. When $\gamma \to \infty$ we obtain the solution (in the sense of a sufficient condition) to conflicts against stationary targets. The guidance law was tested using adjoint and forward simulations for nominal and perturbed interceptor dynamics and for various target maneuvers (step, ramp, and sine functions) and initial heading error. The simulation results are very promising and should encourage further study of the robust guidance law, which showed an obvious superiority over other guidance laws, namely, proportional navigation and minimum effort guidance.

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